Super-Zeeman embedding models on *N*-supersymmetric world-lines

2009 J. Phys. A: Math. Theor. 42 065402

(http://iopscience.iop.org/1751-8121/42/6/065402)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 065402 (12pp) doi:10.1088/1751-8113/42/6/065402

Super-Zeeman embedding models on *N*-supersymmetric world-lines

C F Doran[1], M G Faux[2], S J Gates Jr[3], T Hübsch[4], K M Iga[5] and G D Landweber[6]

[1] Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta T6G 2G1, Canada
[2] Department of Physics, State University of New York, Oneonta, NY 13825, USA
[3] Center for String and Particle Theory, Department of Physics, University of Maryland, College Park, MD 20472, USA
[4] Department of Physics and Astronomy, Howard University, Washington, DC 20059, USA
[5] Natural Science Division, Pepperdine University, Malibu, CA 90263, USA
[6] Department of Mathematics, Bard College, Annandale-on-Hudson, NY 12504-5000, USA

E-mail: doran@math.ualberta.ca, fauxmg@oneonta.edu, gatess@wam.umd.edu, thubsch@howard.edu, Kevin.Iga@pepperdine.edu and gregland@bard.edu

Received 20 October 2008, in final form 1 December 2008
Published 15 January 2009
Online at stacks.iop.org/JPhysA/42/065402

Abstract
We construct a model of an electrically charged magnetic dipole with arbitrary *N*-extended world-line supersymmetry, which exhibits a supersymmetric Zeeman effect. By including supersymmetric constraint terms, the ambient space of the dipole may be tailored into an algebraic variety, and the supersymmetry broken for almost all parameter values. The so-exhibited obstruction to supersymmetry breaking refines the standard one, based on the Witten index alone.

PACS number: 04.65.+e

1. The 1D, $N = 1$ super-Zeeman embedding model

Quantum mechanics with *N*-extended supersymmetry has been a topic of our recurrent interest [1–7]. Here, we construct non-trivial quantum-mechanical models invariant with respect to *arbitrarily high N*-extended supersymmetry, generated by *N* supercharges, Q_I:

$$\{Q_I, Q_J\} = 2\mathrm{i}\delta_{IJ}\partial_\tau, \qquad [Q_I, \partial_\tau] = 0, \qquad I, J = 1, \ldots, N \in \mathbb{N}. \tag{1}$$

In particular, our models exhibit (**1**) coupling to external magnetic fields, (**2**) target space embedding as algebraic varieties and (**3**) supersymmetry breaking by constraint geometry.

To motivate our construction and its generalizations, we first discuss its $N = 1$-supersymmetric toy model version. This model includes *several*, but by no means all the features with which one could endow it. Since our present purpose is the generalization to all $N > 1$, we focus on the select features highlighted above, and defer both other generalizations and most specializations to particular subregions in the multi-dimensional parameter space to a subsequent effort.

A charged particle moving in a two-dimensional, (x, y)-plane may possess

$$\text{angular momentum}: \quad L := m(x\dot{y} - y\dot{x}), \tag{2}$$

$$\text{magnetic dipole moment}: \quad \mu := \frac{q_0}{2mc} L. \tag{3}$$

An external, constant magnetic field $\mathcal{B}_0$ that couples to this magnetic dipole moment contributes to the total energy of this particle through the well-known dipole term, equal to $\mu \mathcal{B}_0 \cos\theta$, where θ is the angle between $\mathcal{B}_0$ and the normal to the (x, y)-plane.

This raises the obvious question: 'is there a generalization of this interaction with an external magnetic field, which is invariant with respect to *arbitrarily N-extended* supersymmetry'?

To answer this question, we start with manifest $N = 1$ supersymmetry, and introduce two real superfields, $\boldsymbol{X}^a$ with $a = 1, 2$, with component fields

$$x^a := \boldsymbol{X}^a| \qquad \text{and} \qquad \chi^a := \mathrm{i} D \boldsymbol{X}^a|, \qquad \text{with} \quad x^1 = x, \quad x^2 = y, \tag{4}$$

where the trailing '|' denotes the evaluation of the preceding superfield expression by setting the Grassmann coordinates to zero, and the factor of i ensures that χ^a too are real. The supersymmetry transformation rules may be written as

$$Q x^a = \chi^a \qquad \text{and} \qquad Q\chi^a = \mathrm{i}\dot{x}^a. \tag{5}$$

With these, we note that the angular momentum, *L*, is part of the 'top' component of a superfield expression:

$$\varepsilon_{ab} D(\boldsymbol{X}^a D\boldsymbol{X}^b)| = -\mathrm{i}\left(\frac{L}{m} - 2\mathrm{i}\chi^1\chi^2\right). \tag{6}$$

As supersymmetry transforms the 'top' component of any superfield expression into a total τ-derivative,

$$\mathscr{L}_L = \mathrm{i}\left(\frac{q_0\mathcal{B}_0}{2c}\cos\theta\right)\varepsilon_{ab} D(\boldsymbol{X}^a D\boldsymbol{X}^b)| = \mu\mathcal{B}_0\cos\theta - 2\mathrm{i}\left(\frac{q_0\mathcal{B}_0}{2c}\cos\theta\right)\chi^1\chi^2 \tag{7}$$

is the $N = 1$ supersymmetrization of the dipole-interaction term, $\mu\mathcal{B}_0\cos\theta$. Before proceeding, we set $c, q_0 \to 1$, so that the Larmor frequency becomes $\omega_L := \frac{q_0\mathcal{B}_0}{mc}\cos\theta \to (\mathcal{B}_0\cos\theta)/m$. We also rescale all fields by $\sqrt{m}$, so that m disappears from the Lagrangian. This fixes the engineering dimensions:

$$[x^a] = [\boldsymbol{X}^a] = -\tfrac{1}{2}, \qquad [\chi^a] = 0 \qquad \text{and} \qquad [\omega_L] = 1, \tag{8}$$

and turns (7) into

$$\mathscr{L}_L = \tfrac{1}{2}\omega_L\varepsilon_{ab}[x^a\dot{x}^b - \mathrm{i}\chi^a\chi^b] = \tfrac{1}{2}\omega_L[(x\dot{y} - y\dot{x}) - 2\mathrm{i}\chi^1\chi^2]. \tag{9}$$

The factor of 2 which appears multiplying $\chi^1\chi^2$ in the final term in (9) may be identified with the Landé g-factor, $g_s = 2$, for spin-$\frac{1}{2}$ particles.

We will also need fermionic superfields $\boldsymbol{\Psi}^A = (\psi^A \mid F^A)$, $A = 0, 1, 2$, with components

$$\psi^A := \boldsymbol{\Psi}^A|, \qquad F^A := D\boldsymbol{\Psi}^A|, \tag{10}$$

the supersymmetry transformations of which may be written as

$$Q\psi^A = \mathrm{i}F^A, \qquad \text{and} \qquad QF^A = \dot{\psi}^A. \tag{11}$$

Here, ψ^A denote fermions and F^A are bosons, and

$$[\boldsymbol{\Psi}^A] = [\psi^A] = 0 \qquad \text{and} \qquad [F^A] = +\tfrac{1}{2}. \tag{12}$$

The toy model Lagrangian for this spinning, charged particle, with $a, b = 1, 2$, is

$$\mathscr{L}_{\text{SZEM}} = \mathscr{L}_B + \mathscr{L}_L + \mathscr{L}_F + \mathscr{L}_{\text{B}\cdot\text{F}} + \mathscr{L}_C, \tag{13a}$$

where the summands are as follows:

$$\mathscr{L}_B = -\tfrac{1}{2}\delta_{ab}D[(D\boldsymbol{X}^a)(D^2\boldsymbol{X}^b)]| = \tfrac{1}{2}\delta_{ab}(\dot{x}^a\dot{x}^b + \mathrm{i}\chi^a\dot{\chi}^b), \tag{13b}$$

$$\mathscr{L}_L = -\tfrac{1}{2}\omega_L\varepsilon_{ab}D[\boldsymbol{X}^a(D\boldsymbol{X}^b)]| = \tfrac{1}{2}\omega_L\varepsilon_{ab}(x^a\dot{x}^b - \mathrm{i}\chi^a\chi^b), \tag{13c}$$

provide the standard kinetic terms for the $(x^a|\chi^a)$ supermultiplets and their ω_L-dependent bilinear interaction term, respectively.

$$\mathscr{L}_F = \tfrac{1}{2}\widehat{\delta}_{AB}D[\boldsymbol{\Psi}^A(D\boldsymbol{\Psi}^B)]| = \tfrac{1}{2}\delta_0(F^0F^0 + \mathrm{i}\psi^0\dot{\psi}^0) + \tfrac{1}{2}\delta_{ab}(F^aF^b + \mathrm{i}\psi^a\dot{\psi}^b), \tag{13d}$$

provides the standard kinetic terms for the $(\psi^A|F^A)$ supermultiplets, and

$$\mathscr{L}_{\text{B}\cdot\text{F}} = \omega_0 D[\delta_{ab}\boldsymbol{\Psi}^a\boldsymbol{X}^b]| = \omega_0\delta_{ab}(F^ax^b + \mathrm{i}\psi^a\chi^b), \tag{13e}$$

provides the ω_0-dependent mixing between $\boldsymbol{X}^a$ and $\boldsymbol{\Psi}^a$. Note that $\boldsymbol{\Psi}^0$ is omitted from $\mathscr{L}_{\text{B}\cdot\text{F}}$, but turns up in

$$\mathscr{L}_C = \tfrac{1}{2}g_0D[\boldsymbol{\Psi}^0(\boldsymbol{X}^ah_{ab}\boldsymbol{X}^b - R^2)]| = g_0\big[\tfrac{1}{2}F^0(x^ah_{ab}x^b - R^2) + \mathrm{i}\psi^0(x^ah_{ab}\chi^b)\big]. \tag{13f}$$

The real parameters occurring in the Lagrangian (13) have the following engineering dimensions:

$$[g_0] = \tfrac{3}{2}, \qquad [\omega_L] = [\omega_0] = 1, \qquad [\delta_0] = [h_{ab}] = 0, \qquad [R] = -\tfrac{1}{2}. \tag{14}$$

The parameters ω_L, ω_0 and g_0 may be used selectively, to turn on/off the Lagrangian terms (13c), (13e) and (13f), and $\delta_0 \to 0$ in (13d) turns $(\psi^0|F^0)$ into Lagrange multipliers.

When $\omega_0, g_0 \to 0$, the $\boldsymbol{X}^a$ decouple from the $\boldsymbol{\Psi}^A$. Their dynamics is governed by $\mathscr{L}_B + \mathscr{L}_L$, as given in equations (13b)–(13c), and $\mathscr{L}_\text{F}$ given in (13d), respectively.

The $\boldsymbol{X}^a$: the Lagrangian (13b)–(13c) produces the coupled equations of motion:

$$\ddot{x} - \omega_L\dot{y} = 0, \qquad \dot{\chi}^1 + \omega_L\chi^2 = 0, \tag{15}$$

$$\ddot{y} + \omega_L\dot{x} = 0, \qquad \dot{\chi}^2 - \omega_L\chi^1 = 0. \tag{16}$$

This result is fairly standard for massive fermions, but not so for bosons: To see this, we 'diagonalize' equations (15)–(16):

$$\big[\partial_\tau^2 + \omega_L^2\big]\partial_\tau x = 0, \qquad \big[\partial_\tau^2 + \omega_L^2\big]\chi^1 = 0, \tag{17}$$

$$\big[\partial_\tau^2 + \omega_L^2\big]\partial_\tau y = 0, \qquad \big[\partial_\tau^2 + \omega_L^2\big]\chi^2 = 0. \tag{18}$$

Indeed, the so-obtained Klein–Gordon equation for χ, η is the standard result, but the corresponding third-order differential equations for the bosons are not. Nevertheless, no unwelcome higher-derivative effect ensues. For any $\omega_L \neq 0$, the solutions of (15)–(16) are

$$x(\tau) = x_+ \cos(\omega_L \tau) + x_- \sin(\omega_L \tau) + x_0, \qquad \chi^1(\tau) = \chi_+ \cos(\omega_L \tau) + \chi_- \sin(\omega_L \tau), \tag{19}$$

$$y(\tau) = -x_- \cos(\omega_L \tau) + x_+ \sin(\omega_L \tau) + y_0, \qquad \chi^2(\tau) = \chi_- \cos(\omega_L \tau) - \chi_+ \sin(\omega_L \tau). \tag{20}$$

This leaves four bosonic, (x_+, x_-, x_0, y_0), and two fermionic (χ_+, χ_-), integration constants to be determined by initial and/or boundary conditions. Now, only x_0 and y_0 parametrize zero modes, the remaining, bosonic–fermionic constant pairs are associated with the ω_L-modes. Thus, the Witten index is[7] $\iota_W(\boldsymbol{X}^a) := (n_B - n_F) = (2 - 0) = 2$.

In the $\omega_L \to 0$ limit, equations (15)–(16) become

$$\ddot{x} = 0, \quad \Rightarrow \quad x(\tau) = v_{x0}\tau + x_0, \qquad \dot{\chi} = 0, \quad \Rightarrow \quad \chi(\tau) = \chi_0, \tag{21}$$

$$\ddot{y} = 0, \quad \Rightarrow \quad y(\tau) = v_{y0}\tau + y_0, \qquad \dot{\eta} = 0, \quad \Rightarrow \quad \eta(\tau) = \eta_0. \tag{22}$$

Since equations (21)–(22) are *uncoupled*, the four bosonic, $(v_{x0}, x_0, v_{y0}, y_0)$, and two fermionic, (χ_0, η_0), integration constants remain independent. The number of *massless* on-shell degrees of freedom (zero modes) then is $n_B = 4$, and $n_F = 2$, leaving the Witten index at $(n_B - n_F) = 2$. While the Witten index remains unchanged in the $\omega_L \to 0$ limit, its separate contributions, n_B and n_F, do change. This follows the original mode-migration wisdom [8], and is presented graphically below.

$$\begin{array}{ccccc} & B & \uparrow & F & \\ & 2 & | & 2 & \\ \iota_W = 2 & 2 & | & 0 & \end{array} \quad \xrightarrow{\omega_L \propto \mathcal{B}_0 \to 0} \quad \begin{array}{ccccc} B & \uparrow & F & \\ 0 & | & 0 & \\ 4 & | & 2 & \iota_W = 2 \end{array} \tag{23}$$

Conversely, turning the magnetic field on, the degeneracy among the zero modes is partially ‘lifted’, producing a supersymmetric Zeeman effect and leaving only two bosonic zero modes.

The $\boldsymbol{\Psi}^A$: the Lagrangian (13*d*) produces the equations of motion $\dot{\psi}^A = 0$ and $F^A = 0$, solved by $d_F + 1$ fermionic constants; see (11). This implies that the Witten index is $\iota_W(\boldsymbol{\Psi}^A) = -3$ if $\delta_0 \neq 0$, and $\iota_W(\boldsymbol{\Psi}^A) = -2$ if we set $\delta_0 = 0$ and drop the $\boldsymbol{\Psi}^0$ superfield.

Using (8) and (12), it is easy to see that the most general mixing and interactions between the fermionic and bosonic supermultiplets $\boldsymbol{X}^a$ and $\boldsymbol{\Psi}^A$ can be introduced via

$$\mathscr{L}_{\text{Int}} = D\{\mathcal{W}(\psi, \boldsymbol{X})\}| = \psi^A \mathcal{W},_A(\psi, x) - \mathrm{i}\chi^a \mathcal{W},_a(\psi, x), \tag{24}$$

where $\mathcal{W}(\psi, \boldsymbol{X})$ is a *fermionic* function of its arguments, and $\mathcal{W},_A$ and $\mathcal{W},_a$ its left-derivatives by ψ^A and $\boldsymbol{X}^a$, respectively. Herein, we focus on the simple, bilinear mixing terms (13*e*).

The combination, $\mathscr{L}_B + \mathscr{L}_L + \mathscr{L}_F + \mathscr{L}_{\text{B-F}}$, as given in (13*b*)–(13*e*), produces the equations of motion, $F^a = -\omega_0 x^a$, the use of which produces

$$\begin{aligned} \mathscr{L}_{B+F}|_{F^a} = {} & \frac{1}{2}\delta_{ab}(\dot{x}^a\dot{x}^b - {\omega_0}^2 x^a x^b) + \frac{1}{2}\omega_L \varepsilon_{ab}(x^a\dot{x}^b - \mathrm{i}\chi^a\chi^b) \\ & + \frac{\mathrm{i}}{2}\delta_{ab}(\chi^a\dot{\chi}^b + \psi^a\dot{\psi}^b) + \mathrm{i}\omega_0\delta_{ab}\psi^a\chi^b. \end{aligned} \tag{25}$$

This describes a two-dimensional, $N = 1$-supersymmetric harmonic oscillator coupled to an external magnetic field. In particular, note that the mixing parameter ω_0 introduced in (13*e*) turns into the (radial) characteristic frequency of this oscillator.

[7] Rotational symmetry implies that an additional bosonic and an additional fermionic mode within (19)–(20) have zero energy. However, they leave the Witten index unchanged, and unless $g_0 = 0$ or $h_{ab} = h\delta_{ab}$, they pair up and acquire nonzero energy. Being interested in the generic properties of the model, we can safely ignore symmetry-related zero modes, which are massless at measure-zero regions in the parameter space.

Higher-dimensional generalizations, with $d > 2$ superfields $\boldsymbol{X}^a, \boldsymbol{\Psi}^b$, will similarly describe supersymmetric, d-dimensional harmonic oscillators coupled to a higher-dimensional external magnetic field, $(\mathscr{F}_0)_{ab}$, replacing the Larmor frequency coefficient, $\omega_L \varepsilon_{ab}$ in (13*e*).

The $\omega_L, \omega_0 = 0$ case: this is the free-field limit where

$$x^a = v_0^a \tau + x_0^a, \qquad \chi^a = \chi_0^a, \qquad \psi^a = \psi_0^a, \qquad a = 1, 2. \tag{26}$$

Counting the bosonic and fermionic constants, the Witten index is, formally, $\iota_W = (4-4) = 0$.

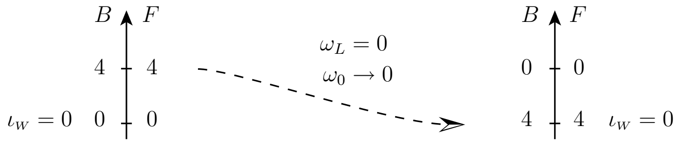
The $\omega_L = 0, \omega_0 \neq 0$ *case:* now, the solutions take the form,

$$x^a = x_+^a \cos(\omega_0 \tau) + x_-^a \sin(\omega_0 \tau), \tag{27}$$

$$\chi^a = \chi_+^a \cos(\omega_0 \tau) + \chi_-^a \sin(\omega_0 \tau), \tag{28}$$

$$\psi^a = \chi_-^a \cos(\omega_0 \tau) - \chi_+^a \sin(\omega_0 \tau), \tag{29}$$

and all constants of integration are associated with modes of nonzero frequency ω_0, so that $\iota_W = (0-0) = 0$. Thus the mode-migration diagram is



(30)

and the difference between (30) and (23) owes to the introduction of the ψ^a fermions.

The $\omega_L, \omega_0 \neq 0$ *case:* the equations of motion are

$$\ddot{x}^a - \omega_L \varepsilon^a{}_b \dot{x}^b + \omega_0{}^2 x^a = 0, \tag{31a}$$

$$\dot{\chi}^a - \omega_L \varepsilon^a{}_b \chi^b - \omega_0 \psi^a = 0, \tag{31b}$$

$$\dot{\psi}^a + \omega_0 \chi^a = 0. \tag{31c}$$

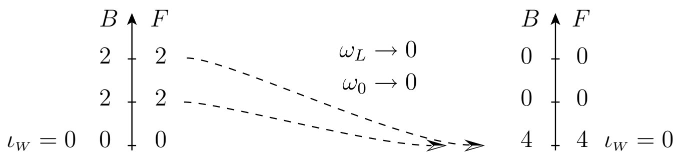
With the help of (31*c*), the time derivative of (31*b*) is

$$\ddot{\chi}^a - \omega_L\, \mathrm{e}^{ab} \delta_{bc} \dot{\chi}^c + \omega_0{}^2 \chi^a = 0, \tag{32}$$

which also follows as the supersymmetry variation of (31*a*). Since (32) is identical in form to (31*a*), so will be the on-shell solutions for x^a and χ^a. It then suffices to discuss only the explicit form of the bosonic solution:

$$x^a(\tau) = A_+^a \cos(\omega_+ \tau) + A_-^a \cos(\omega_- \tau) + \varepsilon^a{}_b A_+^b \sin(\omega_+ \tau) + \varepsilon^a{}_b A_-^b \sin(\omega_- \tau), \tag{33}$$

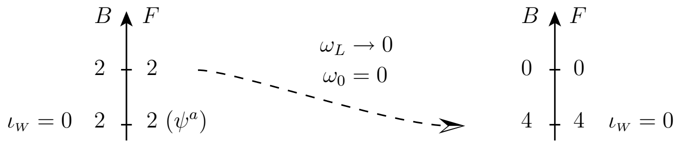
where $\omega_\pm = \frac{1}{2}\left(\sqrt{\omega_L^2 + 4\omega_0^2} \pm \omega_L\right) \geqslant 0$. This solution indicates that there are two constants of integration associated with the frequency ω_+ and two with ω_-. Thus, the mode-migration diagram becomes



(34)

In the first physical quadrant of the (ω_0, ω_L)-plane, the diagram (34) depicts a diagonal path, while (30) follows the ω_0-axis.

On the other hand, along the ω_L-axis we then have



(35)

Note, however, that the $\boldsymbol{\Psi}^a$-modes are *decoupled* from the $\boldsymbol{X}^a$-modes along the $\omega_0 = 0$ edge.

The mode-migration diagrams (30), (34) and (35) show that ω_L partially lifts the degeneracy of the modes, in a Zeeman-like response to an external magnetic field.

The combination of (30), (34) and (35) then covers the behavior in the first quadrant, the physical region, of the (ω_L, ω_0)-plane. The Witten index of the system (25), $\iota_W = 0$, remains constant throughout the physical region of the (ω_0, ω_L) parameter space of this system.

However, in the $\omega_0 = 0 = g_0$ subregion of the parameter space, the system (25) decouples into two separate sub-systems: the $\boldsymbol{X}^a$-system (13*b*)–(13*c*) with $\iota_W(1) = +2$, and the $\boldsymbol{\Psi}^a$-system (13*d*) with $\iota_W(2) = -2$, having set $\delta_0 = 0$ and having dropped $\boldsymbol{\Psi}^0$. As the $g \neq 0$ interactions cannot induce any mode pairing while $\omega_0 = 0$, this obstruction to supersymmetry breaking—*finer* than the overall Witten index [8]—is limited only to this, $\omega_0 = 0$, 'unmixing' edge of the parameter space.

Therefore, any and all supersymmetry breaking effects in the 'bulk' of the whole parameter space must (**1**) vanish in the $\omega_0 \to 0$ limit, and if necessary, (**2**) be discontinuous in this limit.

As far as we know, there are no general guarantees that a supersymmetric model with a non-vanishing Witten index can be embedded into another supersymmetric model with a vanishing Witten index. A result to this effect would seem to be of interest, especially because a nonzero Witten index is understood to obstruct supersymmetry breaking [8].

The obvious embedding of (13*b*)–(13*c*) into (25) is precisely an example of such an embedding.

So, whereas the standard argument [8] prohibits supersymmetry breaking in a model that limits to (13*b*)–(13*c*), the same argument permits supersymmetry breaking in its augmentations that limit to (25)—except at the $\omega_0 = 0$ edge, where the augmentation 'unmixes'.

Since many physics models involve constrained target subspaces, we now return to the full Lagrangian (13*b*)–(13*f*).

Upon eliminating the auxiliary fields F^a, the Lagrangian becomes

$$\widetilde{\mathscr{L}}_{B+F}|_{F^a} = \tfrac{1}{2}\delta_{ab}(\dot{x}^a\dot{x}^b - \omega_0{}^2 x^a x^b) + \tfrac{1}{2}\omega_L\varepsilon_{ab}(x^a\dot{x}^b - \mathrm{i}\chi^a\chi^b) + \tfrac{1}{2}\mathrm{i}\delta_{ab}(\chi^a\dot{\chi}^b + \psi^a\dot{\psi}^b) \\ + \mathrm{i}\omega_0\delta_{ab}\psi^a\chi^b + \tfrac{1}{2}g_0F^0(x^a h_{ab}x^b - R^2) + \mathrm{i}g_0\psi^0(x^a h_{ab}\chi^b), \qquad (36)$$

and the equations of motion now become

$$\ddot{x}^a - \omega_L\varepsilon^a{}_b\dot{x}^b + \big(\omega_0{}^2\delta^a{}_b - g_0F^0h^a{}_b\big)x^b - \mathrm{i}g_0\psi^0h^a{}_b\chi^b = 0, \qquad (37)$$

$$\dot{\chi}^a - \omega_L\varepsilon^a{}_b\chi^b - \omega_0\psi^a - g_0\psi^0h^a{}_bx^b = 0, \qquad (38)$$

$$\dot{\psi}^a + \omega_0\chi^a = 0, \qquad (39)$$

$$x^a h_{ab}x^b - R^2 = 0, \qquad (40)$$

$$x^a h_{ab}\chi^b = 0, \qquad (41)$$

where $\varepsilon^a{}_b := \delta^{ac}\varepsilon_{cb}$ and $h^a{}_b := \delta^{ac}h_{cb}$.

Constraint geometry: (40) implies that $\|x\|_h^2 = R^2$, constraining the two bosons to this quadratic curve $\mathscr{C} \subset \mathbb{R}^2$, the geometry of which is controlled by h_{ab}: $\mathscr{C}$ is an ellipse if h_{ab} is positive definite, a hyperbola if h_{ab} has eigenvalues of both signs, and a straight line if one of the eigenvalues vanishes and the other is positive. Otherwise, $\mathscr{C} = \varnothing$. Equation (41) implies that χ^b are h_{ab}-orthogonal to x^a, i.e., tangential to $\mathscr{C}$ at each of its points. Thus, the χ^b span the fibers of the tangent bundle, $T_{\mathscr{C}}$.

Dynamics: one of the two equations (37) may be used to express F^0 in terms of the other fields; the remaining equation governs the dynamics of the x^a, constrained to $\mathscr{C}$. One of the two equations (38) may be used to express ψ^0 in terms of the other fields, and the remaining equation governs the dynamics of the χ^a, constrained to the fibers of $T_{\mathscr{C}}$. Finally, equations (39) relate a combination of the ψ^a's to a corresponding $\chi^a|_{T_{\mathscr{C}}}$ as a matching pair to (38), while the other combination of the ψ^a's is restricted from varying away from $T^*_{\mathscr{C}}$.

Example: by selecting $h_{22} = 1$ to be the only nonzero element of h_{ab}, we obtain

$$x^2 = \pm R, \qquad \chi^2 = 0, \qquad F^0 = \frac{\pm 1}{g_0 R}(\omega_L \dot{x}^1 \pm {\omega_0}^2 R), \qquad \psi^0 = \frac{\pm 1}{g_0 R}(\omega_L \chi^1 - \omega_0 \psi_0^2), \tag{42}$$

$$\ddot{x}^1 + {\omega_0}^2 x^1 = 0, \qquad \dot{\chi}^1 - \omega_0 \psi^1 = 0 = \dot{\psi}^1 + \omega_0 \chi^1, \qquad \psi^2 = \text{const.} \tag{43}$$

Here $\mathscr{C}$ consists of two copies of the $x^1 \equiv x$-axis, positioned at $x^2 \equiv y = \pm R$. The fermions $\chi^1 \equiv \chi$ and ψ^1 span $T_{\mathscr{C}}$ and $T^*_{\mathscr{C}}$, respectively[8], $\chi^2 = 0$, $\psi^2 = \text{const.}$, and F^0, ψ^0 are functions of other fields.

Supersymmetry breaking: the bosonic potential is obtained from the negative of (36), by setting all fermions and all τ-derivatives to zero, and enforcing the constraint (40):

$$\mathscr{V}(x|_{\mathscr{C}}) = \tfrac{1}{2}{\omega_0}^2 (x^a \delta_{ab} x^b)|_{\|x\|_h^2 = R^2}. \tag{44}$$

Now, if we choose $h_{ab} = h\delta_{ab}$, the constraint (40) implies that

$$\mathscr{V}(x|_{\mathscr{C}}) = \tfrac{1}{2}{\omega_0}^2 R^2/h > 0, \quad \text{since} \quad \text{sign}(h) \overset{(40)}{=} \text{sign}(R^2). \tag{45}$$

This precludes the total energy of the system from vanishing, and so also the existence of supersymmetric ground states: supersymmetry is spontaneously broken. The same holds for other choices of h_{ab}, regardless of its (in)definiteness, as long as $R \neq 0$. In fact, even in the analytic continuation to $R^2 < 0$ in the manner of Witten [9], the result (45) remains true and $R \neq 0$ breaks supersymmetry.

Recall now that the $\omega_0 = 0$ edge of the physical parameter space harbors the formal obstruction to supersymmetry breaking. Indeed, $\lim_{\omega_0 \to 0} \mathscr{V}(x|_{\mathscr{C}}) = 0$, and supersymmetry breaking is turned off in the $\omega_0 \to 0$ limit. The same is true in the $R \to 0$ limit.

In turn, neither $\mathscr{V}(x)$ nor $\mathscr{V}(x|_{\mathscr{C}})$ depends on ω_L: the coupling to the external magnetic field has no effect on supersymmetry breaking, and the Zeeman effect is supersymmetric.

The effect of $\delta_0 \neq 0$ is that ψ^0 also becomes a dynamical fermion, while F^0 still has a purely algebraic equation of motion: $F^0 = -\frac{1}{2}(g_0/\delta_0)\big(\|x\|_h^2 - R^2\big)$. Upon substituting this back into the Lagrangian, the potential becomes

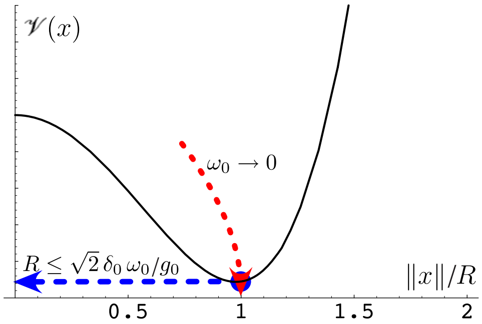
$$\mathscr{V}(x) = \frac{1}{2}{\omega_0}^2 (x^a \delta_{ab} x^b) + \frac{1}{8}\left(\frac{g_0}{\delta_0}\right)^2 ((x^a h_{ab} x^b) - R^2)^2. \tag{46}$$

When $h_{ab} = \delta_{ab}$, the extrema of this include the origin, $x^a = 0$, and the circle of radius $\|x\| = \sqrt{R^2 - 2(\delta_0\omega_0/g_0)^2}$ when $R \geqslant \sqrt{2}\delta_0\omega_0/g_0$, breaking the $SO(2)$ symmetry[9].

[8] In fact, the supersymmetry transformations also imply this: $Q : x^a \to \chi^a$, whereas $Q : \psi^a \to F^a = -\omega_0 x^a$.

[9] We forego gauging this as well as other symmetries that emerge in special regions of the parameter space.

Throughout, however, $\mathscr{V}(x)$ remains positive—*signaling supersymmetry breaking*—except when $\omega_0 \to 0$. The graph



Supersymmetry is restored in the $\omega_0 \to 0$ limit.

$SO(2)$rotational symmetry is restored when $R \leqslant \sqrt{2}\delta_0\omega_0/g_0$. (47)

shows the effect of the variation of ω_0 and R on the potential $\mathscr{V}(x)$; see (46). This role of $\omega_0 \to 0$ as the supersymmetry restoration limit is perfectly in agreement with the above analysis of the obstruction to supersymmetry breaking.

From equations (45), (46) and the graph (47), we see that

(1) the choice of h_{ab} controls the geometry of
   (a) the constrained subspace when $\delta_0 = 0$, as illustrated in equations (42)–(43),
   (b) the minimum of the potential (46) when $\delta_0 \neq 0$,

(2) for nonzero h_{ab},
   (a) R controls the size of the constrained subspace, i.e., potential minimum,
   (b) the product $(\omega_0 R) \neq 0$ signals supersymmetry breaking.

2. $N > 1$ isoscalar and isospinor supermultiplets

With the foregoing analysis of the $N = 1$ case, we now turn to the much more interesting generalization to *arbitrary* $N > 1$.

The transformation rules of the *isoscalar* supermultiplet [10] may be written as

$$Q_I x_i = (L_I)_i{}^{\hat{j}} \chi_{\hat{j}}, \qquad Q_I \chi_{\hat{\imath}} = i(R_I)_{\hat{\imath}}{}^{j} \dot{x}_j, \tag{48}$$

which straightforwardly generalizes (5). Although the number of bosonic and fermionic component fields is the same, 2^{N-1}, we find it useful to distinguish between the indices, i versus $\hat{\imath}$, that count them. Finally, we note that such a supermultiplet has the *topology* [7] of the N-cube, $[0, 1]^N$. Smaller *quotient* supermultiplets may be obtained using certain *projections*, as classified in [11]. Including also these quotient supermultiplets provides an extension of our present analysis, but is beyond our present scope.

For consistency with (1), the $\mathbb{L}_I$ and $\mathbb{R}_I$ matrices in (48) must satisfy

$$(L_I)_i{}^{\hat{j}}(R_J)_{\hat{j}}{}^{k} + (L_J)_i{}^{\hat{j}}(R_I)_{\hat{j}}{}^{k} = 2\delta_{IJ}\delta_i{}^{k}, \quad \text{i.e.,} \quad \mathbb{L}_I\mathbb{R}_J + \mathbb{L}_J\mathbb{R}_I = 2\delta_{IJ}\mathbb{1}, \tag{49a}$$

$$(R_J)_{\hat{\imath}}{}^{j}(L_I)_j{}^{\hat{k}} + (R_I)_{\hat{\imath}}{}^{j}(L_J)_j{}^{\hat{k}} = 2\delta_{IJ}\delta_{\hat{\imath}}{}^{\hat{k}}, \quad \text{i.e.,} \quad \mathbb{R}_I\mathbb{L}_J + \mathbb{R}_J\mathbb{L}_I = 2\delta_{IJ}\mathbb{1}. \tag{49b}$$

The $J = I$ cases then imply that

$$\left.\begin{array}{ll} (L_I)_i{}^{\hat{j}}(R_I)_{\hat{j}}{}^{k} & = \delta_i{}^{k} \\ (R_I)_{\hat{\imath}}{}^{j}(L_I)_j{}^{\hat{k}} & = \delta_{\hat{\imath}}{}^{\hat{k}} \end{array}\right\} \quad \text{i.e.,} \qquad \mathbb{R}_I = \mathbb{L}_I^{-1}, \quad I = 1, \ldots, N. \tag{50}$$

Generalizing similarly (11), we introduce *isospinor* supermultiplets, $\mathbf{\Psi}^A = \left(\psi_{\hat{\imath}}^A, F_i^A\right)$:

$$Q_I \psi_{\hat{\imath}} = \mathrm{i}(R_I)_{\hat{\imath}}{}^j F_j, \qquad Q_I F_i = (L_I)_i{}^{\hat{\jmath}} \dot{\psi}_{\hat{\jmath}}. \tag{51}$$

Supersymmetry of the standard kinetic terms, generalizing (13*b*), implies the relation[10],

$$\mathbb{R}_I = \mathbb{L}_I^T, \quad \text{i.e.} \quad (R_I)_{\hat{\jmath}}{}^k \delta_{ik} = (L_I)_i{}^{\hat{k}} \delta_{\hat{\jmath}\hat{k}}. \tag{52}$$

The conditions in (49), (50) and (52) define the $\mathcal{GR}(\mathrm{d}, \mathcal{N})$ 'Garden' algebras introduced in [2, 3].

For $N > 1$, these are *not* the familiar Salam–Strathdee superfields of N-extended supersymmetry, although theorem 7.6 in [7] relates them[11]. In particular, in the isoscalar supermultiplet $\boldsymbol{X}^a$, all bosons x_i^a have the same engineering dimension, $\frac{1}{2}$ less than the fermions, $\chi_{\hat{\imath}}^a$. Consequently, given a suitable Lagrangian, all bosons and all fermions are physical, propagating component fields. There are neither auxiliary nor gauge degrees of freedom in $\boldsymbol{X}^a$; this is also true of its quotient supermultiplets, mentioned above.

Generalizing (13*c*), we seek an N-supersymmetric term of the general form,

$$\mathscr{L}_L = \tfrac{1}{2}\left\{\omega_{ab}^{ij} x_i^a \dot{x}_j^b - \mathrm{i}\widehat{\omega}_{ab}^{\hat{\imath}\hat{\jmath}} \chi_{\hat{\imath}}^a \chi_{\hat{\jmath}}^b\right\}, \tag{53}$$

with

$$\omega_{ab}^{ij} = -\omega_{ba}^{ji} \qquad \text{and} \qquad \widehat{\omega}_{ab}^{\hat{\imath}\hat{\jmath}} = -\widehat{\omega}_{ba}^{\hat{\jmath}\hat{\imath}}, \tag{54}$$

and require that it be invariant with respect to the supersymetry transformation (48). This is the case if and only if

$$(L_I)_i{}^{\hat{k}} \omega_{ab}^{i\ell} = \widehat{\omega}_{ab}^{\hat{k}\hat{\jmath}} (R_I)_{\hat{\jmath}}{}^{\ell}, \qquad \text{for all } a, b, \ell, \hat{k}, I. \tag{55}$$

Condition (55) simplifies in the special case, when there is an even number, $d = 2p$, supermultiplets $\boldsymbol{X}^a$. Then, we divide the supermultiplets $\boldsymbol{X}^a \to \boldsymbol{X}^{\alpha\hat{a}}$ into pairs, so that $\alpha = 1, 2$ and $\hat{a} = \left\lfloor \frac{a+1}{2} \right\rfloor = 1, \ldots, p$. The properties (54) are then satisfied by choosing

$$\omega_{ab}^{ij} = \varepsilon_{\alpha\beta} \delta^{ij} \omega_{(\hat{a}\hat{b})} \qquad \text{and} \qquad \widehat{\omega}_{ab}^{\hat{\imath}\hat{\jmath}} = \varepsilon_{\alpha\beta} \delta^{\hat{\imath}\hat{\jmath}} \widehat{\omega}_{(\hat{a}\hat{b})}. \tag{56}$$

Setting then, in addition, $\omega_{(\hat{a}\hat{b})} = \widehat{\omega}_{(\hat{a}\hat{b})}$, with (56) ensures the supersymmetry of the Lagrangian (53).

Together with the standard kinetic terms, this produces

$$\mathscr{L}_B + \mathscr{L}_L = \tfrac{1}{2}\delta_{\hat{a}\hat{b}}\delta_{\alpha\beta}\left\{\delta^{ij}\dot{x}_i^{\alpha\hat{a}}\dot{x}_j^{\beta\hat{b}} + \mathrm{i}\delta^{\hat{\imath}\hat{\jmath}}\chi_{\hat{\imath}}^{\alpha\hat{a}}\dot{\chi}_{\hat{\jmath}}^{\beta\hat{b}}\right\} + \tfrac{1}{2}\omega_{(\hat{a}\hat{b})}\left\{\delta^{ij}\left(x_i^{1\hat{a}}\dot{x}_j^{2\hat{b}} - \dot{x}_i^{1\hat{a}}x_j^{2\hat{b}}\right) - 2\mathrm{i}\delta^{\hat{\imath}\hat{\jmath}}\chi_{\hat{\imath}}^{1\hat{a}}\chi_{\hat{\jmath}}^{2\hat{b}}\right\}, \tag{57}$$

generalizing (13*b*)–(13*c*).

The structure of the matrix of Larmor frequencies (56) implies that the supermultiplets $\boldsymbol{X}^{\alpha\hat{a}} = \left(x_i^{\alpha\hat{a}} \middle| \chi_i^{\alpha\hat{a}}\right)$ are organized into p pairs, $(\boldsymbol{X}^{1\hat{a}}, \boldsymbol{X}^{2\hat{a}})$. Each $\boldsymbol{X}^{1\hat{a}}$ then mixes with each $\boldsymbol{X}^{2\hat{b}}$ by an amount controlled by the Larmor frequency $\omega_{(\hat{a}\hat{b})}$. In turn, the 'trace' angular momentum,

$$L^{(\hat{a}\hat{b})} := \delta^{ij}\left(x_i^{1\hat{a}}\dot{x}_j^{2\hat{b}} - \dot{x}_i^{1\hat{a}}x_j^{2\hat{b}}\right), \tag{58}$$

couples, with the strength of $\omega_{(\hat{a}\hat{b})}$, to the external magnetic field through the $\left(x_i^{1\hat{a}}, x_i^{2\hat{b}}\right)$-planes.

We finally arrive at

$$\mathscr{L}_{\text{SZEM}} = \mathscr{L}_B + \mathscr{L}_L + \mathscr{L}_F + \mathscr{L}_{\text{B}\cdot\text{F}} + \mathscr{L}_C, \tag{59}$$

[10] The difference with respect to the original relation [2, 3] owes to an overall sign convention. Our present convention keeps the forms of (48), (51) in the $N \to 1$ limit, and so also that of corresponding Lagrangians.

[11] Reference [10] translates the standard kinetic terms into superfield notation. We see no obstruction to doing so also for our complete Lagrangian (59).

where $\mathscr{L}_B + \mathscr{L}_L$ are given in (57), and

$$\mathscr{L}_F = \tfrac{1}{2}\delta_0\big(\delta^{ij}F_i^0F_j^0 + \mathrm{i}\delta^{\hat{\imath}\hat{\jmath}}\psi_{\hat{\imath}}^0\dot{\psi}_{\hat{\jmath}}^0\big) + \tfrac{1}{2}\delta_{ab}\big(\delta^{ij}F_i^aF_j^b + \mathrm{i}\delta^{\hat{\imath}\hat{\jmath}}\psi_{\hat{\imath}}^a\dot{\psi}_{\hat{\jmath}}^b\big), \tag{60}$$

$$\mathscr{L}_{\mathrm{B\cdot F}} = \omega_0\delta_{ab}\big(\delta^{ij}F_i^ax_j^b + \mathrm{i}\delta^{\hat{\imath}\hat{\jmath}}\psi_{\hat{\imath}}^a\chi_{\hat{\jmath}}^b\big), \tag{61}$$

$$\mathscr{L}_C = g_0\big\{\tfrac{1}{2}F_i^0\big(h_{ab}^{ijk}x_j^ax_k^b - h^iR^2\big) + \mathrm{i}\psi_{\hat{\imath}}^0\big(\hat{h}_{ab}^{\hat{\imath}j\hat{k}}x_j^a\chi_{\hat{k}}^b\big)\big\}. \tag{62}$$

Supersymmetry: owing to the fact that each auxiliary field F_i^A transforms into a total τ-derivative, the linear term $\frac{1}{2}g_0F_i^0h^iR^2$ is supersymmetric all by itself for any dimensionless 2^{N-1}-vector, h^i.

By construction of (62),

$$h_{ab}^{ijk} = h_{ba}^{ikj}. \tag{63a}$$

The arbitrary N-extended supersymmetry of (62) is then ensured if the arrays of dimensionless constants h_{ab}^{ijk} and $\hat{h}_{ab}^{\hat{\imath}j\hat{k}}$ satisfy

$$(R_I)_{\hat{\ell}}{}^i\hat{h}_{ab}^{\hat{\ell}j\hat{k}} = h_{ab}^{ijm}(L_I)_m{}^{\hat{k}}, \qquad \hat{h}_{ab}^{\hat{\imath}j\hat{\ell}}(R_I)_{\hat{\ell}}{}^k = (L_I)_\ell{}^{\hat{\imath}}h_{ab}^{\ell jk}, \tag{63b}$$

$$\hat{h}_{ab}^{\hat{\imath}\ell\hat{k}}(L_I)_\ell{}^{\hat{\jmath}} = \hat{h}_{ba}^{\hat{\imath}\ell\hat{\jmath}}(L_I)_\ell{}^{\hat{k}}. \tag{63c}$$

Using (52), equations (63b) produce

$$\hat{h}_{ab}^{\hat{\imath}j\hat{k}} = \frac{1}{N}\sum_{I=1}^{N}(L_I)_\ell{}^{\hat{\imath}}h_{ab}^{\ell jm}(L_I)_m{}^{\hat{k}}, \tag{64}$$

which may be used as a *definition* of $\hat{h}_{ab}^{\hat{\imath}\hat{\jmath}\hat{k}}$ in terms of h_{ab}^{ijk}. In fact, conditions (63b)–(63c) are all satisfied if the N contributions in the defining sum (64) are all identical.

As the system of constraints (63) may seem over-constraining, we exhibit the simplest non-trivial ansatz to solve conditions (63) for $N = 2$, where we choose

$$\mathbb{L}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \mathbb{R}_1, \qquad \mathbb{L}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \mathbb{R}_2, \tag{65}$$

restrict $a, b = 1, 2$, and find

$$\big(h_{11}^{ijk}\big) = \left(\begin{bmatrix} A & B \\ B & -A \end{bmatrix}, \begin{bmatrix} -B & A \\ A & B \end{bmatrix}\right), \quad \Rightarrow \quad \big(\hat{h}_{11}^{\hat{\imath}j\hat{k}}\big) = \left(\begin{bmatrix} A & -B \\ B & A \end{bmatrix}, \begin{bmatrix} B & A \\ -A & B \end{bmatrix}\right), \tag{66}$$

$$= \delta^{j1}(A\delta^{ik} + B\varepsilon^{ik}) + \delta^{j2}(B\delta^{ik} - A\varepsilon^{ik}), \tag{67}$$

where A, B are two arbitrary constants; the index i labels the two blocks, in which j, k are the row and column indices, respectively. $\big(h_{12}^{ijk}\big)$ and $\big(\hat{h}_{12}^{\hat{\imath}j\hat{k}}\big)$ are of the same form, depending on another two arbitrary constants, and so are $\big(h_{22}^{ijk}\big)$ and $\big(\hat{h}_{22}^{\hat{\imath}j\hat{k}}\big)$.

Note now that the $N = 1$ constraint Lagrangian (13f) depended on three constants, h_{ab}, which is, for the $N = 2$ constraint system (62), generalized to the six constants in h_{ab}^{ijk}. These numbers precisely fit the generic formula that one would expect, $N \cdot \binom{d+1}{2}$, where N stems from the N-extendedness of supersymmetry, and $\binom{d+1}{2}$ is the number of parameters in a quadratic[12]

[12] For degree-q polynomials, $\big(h_{a_1\ldots b_q}^{ij_1\ldots j_q}\big)$ subject to a generalization of (63) and (68) would depend on $N \cdot \binom{d+q-1}{q}$ parameters; effectively, one degree-q polynomial in d coordinates for each of N supersymmetries.

polynomial in d coordinates. We thus expect no obstruction to finding solutions to conditions (63) for arbitrary N and d.

Looking at the solution above and using the representation theory of the $\mathcal{GR}(\mathrm{d},\mathcal{N})$ algebras [4], it seems practical to expect that there exist constant coefficients $h_{ab\ell}$ such that

$$\big(h^{ijk}_{ab}\big) = \sum_r \big(h_{ab\ell}(f^{I_1\ldots I_{2r}})^{\ell j}(f_{I_1\ldots I_{2r}})^{ik} + h_{ba\ell}(f^{I_1\ldots I_{2r}})^{\ell k}(f_{I_1\ldots I_{2r}})^{ij}\big), \tag{68}$$

with the Clebsch–Gordan-like coefficients $(f_{I_1\ldots I_{2\ell}})^{ik}$ defined in [3], is the natural and covariant generalization of the $N = 2$ specific result (66)–(67). Though additional computations remain to construct explicit such models for even higher values of N and for which the conditions in (63) are satisfied, this method does seem to have the potential to open a new study on whether it is possible to construct a model with $N \to N'$-extended supersymmetry breaking, with $N > N' \neq 0$.

The $\delta_0 = 0$ case: as in the $N = 1$ case, the component fields of the supermultiplet $\big(\psi^0_i \big| F^0_i\big)$ act as Lagrange multipliers. The auxiliary fields F^0_i enforce the N bosonic constraints:

$$h^{ijk}_{ab}x^a_j x^b_k = h^i R^2, \tag{69}$$

whereas the fermions, ψ^0_i, impose Nh-orthogonality constraints:

$$\hat{h}^{\hat{\imath}j\hat{k}}_{ab}x^a_j\chi^b_{\hat{k}} = 0. \tag{70}$$

A detailed analysis of the geometry of the so-defined target space is beyond our present scope, but it should be clear that the system (69)–(70), parametrized by the $N\cdot\binom{d+1}{2}$ parameters in h^{ijk}_{ab} and the 2^{N-1} components of h^i, offers considerable choices.

The $\delta_0 \neq 0$ *case:* the equations of motion for the fields, F^0_i, now become

$$F^0_i = -\frac{g_0}{2\delta_0}\delta_{ij}\big(h^{jk\ell}_{ab}x^a_k x^b_\ell - h^j R^2\big), \tag{71}$$

which, when substituted back into the Lagrangian produces the potential

$$\mathscr{V}(x) = \frac{1}{2}{\omega_0}^2\big(x^a_i\delta^{ij}\delta_{ab}x^b_j\big) + \frac{1}{8}\left(\frac{g_0}{\delta_0}\right)^2\big\|\big(h^{ijk}_{ab}x^a_j x^b_k - h^i R^2\big)\big\|^2, \tag{72}$$

providing the $N > 1$ and d-dimensional generalization of (46).

From equations (69) and (72), we see that the coefficients h^{ijk}_{ab} assume the rôle of h_{ab} in equations (45) and (46), and control the geometry of the constrained subspace, i.e. the minimum of the potential.

3. Conclusions

We have shown that it is possible to construct a new class of supersymmetical quantum-mechanical models that admit an *arbitrary number of supercharges* and interactions with static, background magnetic fields (i.e., fluxes). In the case of $N = 1$, we have seen how the mode migration of the model is in accord with the Witten index arguments [8], but have also found that in a certain limit there is an obstruction to supersymmetry breaking that is finer than that based on the (overall) index alone. For the case of $N = 2$, we have also introduced explicit interactions, including some of a form very similar to Landau–Ginzburg models involving toric geometry [9]. We also provided a parameter-counting argument to indicate that, in fact, there is no obstruction to constructing the model (59) for any $N \in \mathbb{N}$. Our detailed analysis of the $N = 1$ model, embedded within the $N = 2$ one, implies that, for all N, the bulk of the $\left\{\delta_0, \omega_0, g_0, R, h^i, h_{ab}^{ijk}\right\}$ parameter space describes a phase where supersymmetry is spontaneously broken, but is restored in the portion of the boundary where $\omega_0 R = 0$.

The current study does not exhaust the class of models with interactions and arbitrary numbers of supersymmetries that may be constructed as generalizations of these techniques. In particular, in future efforts it seems indicated that the special case of $N = 32$ may be of special interest, at which point the classification of [11] will help with the combinatorial complexity.

The two pressing questions are whether such a model can be constructed that might allow a new method for the study of *M*-theory, and whether the constrained target space may be tailored into a Hořava–Witten spacetime with boundary brane-worlds [12, 13], complete with a Randall–Sundrum geometry [14, 15].

You cannot depend on your eyes when your imagination is out of focus.
Mark Twain

Acknowledgments

The research of SJG is supported in part by the National Science Foundation Grant PHY-0354401, the endowment of the John S Toll Professorship and the CSPT. TH is indebted to the support of the Department of Energy through the grant DE-FG02-94ER-40854.

References

[1] Gates S J Jr and Rana L 1995 Ultramultiplets: a new representation of rigid 2-d, $N = 8$ supersymmetry *Phys. Lett.* B **342** 132–7 (arXiv:hep-th/9410150)
[2] Gates S J Jr and Rana L 1995 A theory of spinning particles for large *N*-extended supersymmetry *Phys. Lett.* B **352** 50–8 (arXiv:hep-th/9504025)
[3] Gates S J Jr and Rana L 1996 A theory of spinning particles for large *N*-extended supersymmetry: II. *Phys. Lett.* B **369** 262–8 (arXiv:hep-th/9510151)
[4] Gates S J Jr, Linch W, Phillips J and Rana L 2002 The fundamental supersymmetry challenge remains *Grav. Cosmol.* **8** 96–100 (arXiv:hep-th/0109109)
[5] Gates S J Jr, Linch III W D and Phillips J 2002 When superspace is not enough arXiv:hep-th/0211034
[6] Faux M and Gates S J Jr 2005 Adinkras: a graphical technology for supersymmetric representation theory *Phys. Rev.* D **71** 065002 (arXiv:hep-th/0408004)
[7] Doran C F, Faux M G, Gates S J Jr, Hübsch T, Iga K M and Landweber G D 2007 On graph-theoretic identifications of Adinkras, supersymmetry representations and superfields *Int. J. Mod. Phys.* A **22** 869–930 (arXiv:math-ph/0512016)
[8] Witten E 1982 Constraints on supersymmetry breaking *Nucl. Phys.* B **202** 253
[9] Witten E 1993 Phases of $N = 2$ theories in two-dimensions *Nucl. Phys.* B **403** 159–222
[10] Doran C F, Faux M G, Gates S J Jr, Hübsch T, Iga K M and Landweber G D 2008 Adinkras and the dynamics of superspace prepotentials *Adv. S. Th. Phys.* **2** 113–64 (arXiv:hep-th/0605269)
[11] Doran C F, Faux M G, Gates S J Jr, Hübsch T, Iga K M, Landweber G D and Miller R L 2008 Topology types of Adinkras and the corresponding representations of *N*-extended supersymmetry arXiv:0806.0050
[12] Hořava P and Witten E 1996 Heterotic and type-I string dynamics from eleven dimensions *Nucl. Phys.* B **460** 506–24 (arXiv:hep-th/9510209)
[13] Hořava P and Witten E 1996 Eleven-dimensional supergravity on a manifold with boundary *Nucl. Phys.* B **475** 94–114 (arXiv:hep-th/9603142)
[14] Randall L and Sundrum R 1999 A large mass hierarchy from a small extra dimension *Phys. Rev. Lett.* **83** 3370–3 (arXiv:hep-ph/9905221)
[15] Randall L and Sundrum R 1999 An alternative to compactification *Phys. Rev. Lett.* **83** 4690–3 (arXiv:hep-th/9906064)